Assignment 1

1. Let f be a 2π-periodic function which is integrable over $[-\pi, \pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx = \int_J f(x)dx,$$
where I and J are intervals of length 2π.

2. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

3. Here all functions are defined on $[-\pi, \pi]$. Verify their Fourier expansion and determine their convergence and uniform convergence (if possible).

   (a)
$$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx,$$

   (b)
$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

   (c)
$$f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$$

   (d)
$$g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

4. Show that
$$x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty} \frac{\sin nx}{n},$$
for $x \in [0, 2\pi]$. Compare it with 1(a).

5. A finite Fourier series is of the form $a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$. A trigonometric polynomial is of the form $p(\cos x, \sin x)$ where $p(x, y)$ is a polynomial of two variables x, y. Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

6. This is an optional problem.

   (a) Assume that the Fourier coefficients of a continuous, 2π-periodic function vanish identically. Show that this function must be the zero function. Hint: WLOG assume $f(0) > 0$. Use the relation
$$\int_{-\pi}^{\pi} f(x)p(x)dx = 0,$$
where $p(x)$ is a trigonometric polynomial of the form $(\varepsilon + \cos x)^k$ for some small ε and large $k > 0$.

(b) Use the result in (a) to show that if the Fourier series of a continuous, 2π-periodic function converges uniformly, then it converges uniformly to the function itself.

(c) Apply (b) to Problem 1(a) to show

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$